

Zeroes of Real Valued Eigenfunctions

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Suppose we are given a symmetric operator T acting on a subspace of $L_2(M_n, \mu)$ where M_n is an n -dimensional metrizable connected manifold and μ is a measure that is positive on open sets in M_n . Then there is at most one eigenspace that contains a real valued eigenfunction whose nodal set has dimension less than $n - 1$.

Spectral Theory (math.ST)

Let M_n be a connected n -dimensional metrizable manifold and let μ be a measure on M_n that is positive on open sets. Suppose that T is a symmetric operator acting on a subspace of $L_2(M_n, \mu)$.

Theorem. *There is at most one eigenspace that contains a real valued continuous function ψ for which the nodal set N_ψ of its zeroes has dimension less than $n - 1$.*

Proof: First we will need the following.

Lemma. *If ψ is any continuous real valued function on M_n whose nodal set N_ψ has dimension less than $n - 1$, then ψ is of constant sign on the complement of N_ψ*

Proof of lemma: The complement of N_ψ is connected [1] and $\psi(x) \neq 0$ for any x outside N_ψ . This establishes the lemma.

Next, let ψ_1 and ψ_2 be real valued continuous eigenfunctions corresponding to different eigenvalues, and suppose that the dimensions of N_{ψ_1} and N_{ψ_2} are each less than $n - 1$. Since $N_{\psi_1\psi_2} = N_{\psi_1} \cup N_{\psi_2}$, it follows from the sum theorem for dimension that the dimension of $N_{\psi_1\psi_2}$ is less than $n - 1$. Our lemma implies that either $\psi_1\psi_2$ or $-\psi_1\psi_2$ is positive outside $N_{\psi_1\psi_2}$. It follows that the integral of $\psi_1\psi_2$ over M_n cannot be zero, which gives us a contradiction and establishes our theorem.

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We will say that an eigenvalue λ for which there is a ψ in the corresponding eigenspace whose nodal set N_ψ is of dimension less than $n - 1$ is an *exceptional eigenvalue*.

Corollary. *If M_n and ψ are real analytic and the eigenvalue for ψ is not exceptional, then the dimension of N_ψ must equal $n - 1$.*

Proof: If the dimension of N_ψ equalled n , then N_ψ would have to contain an open set[1].

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References

- [1] Hurewicz, Witold and Wallman, Henry, *Dimension Theory*, Princeton University Press, 1969.